Dissection of rhombic solids with octahedral symmetry to Archimedean solids, Part 1

Izidor Hafner

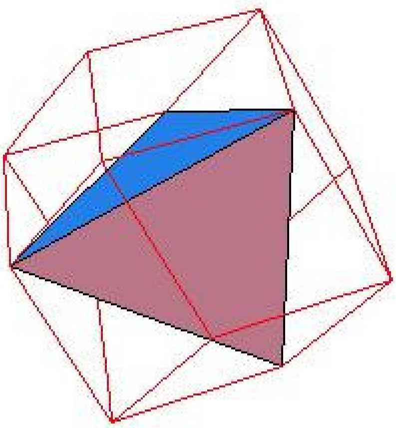
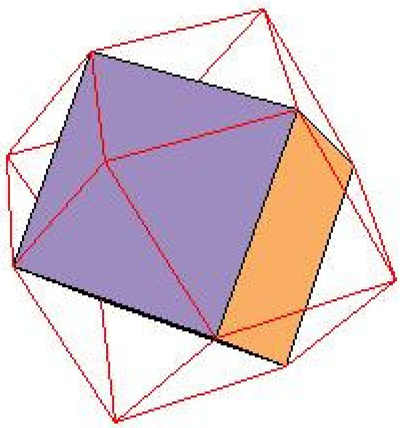
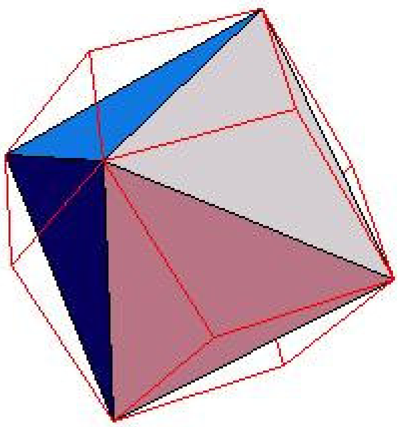
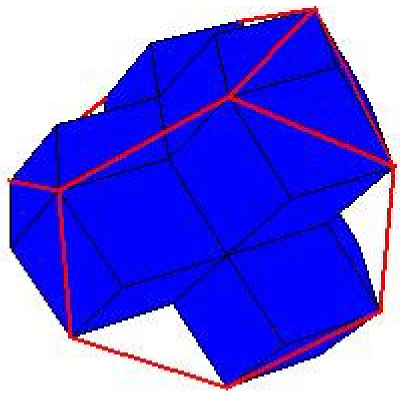
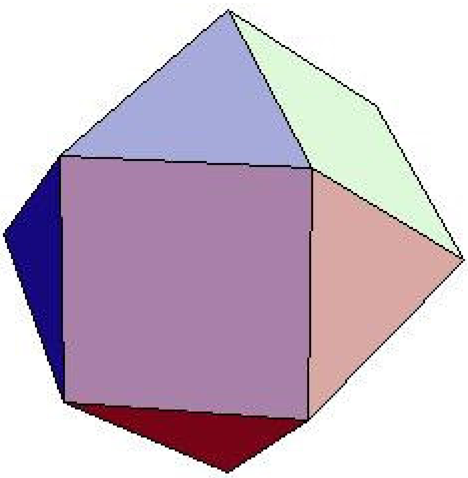
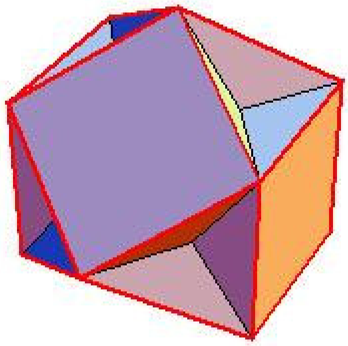
Our result in [5] shows that a rhombic 36-hedron, a solid composed by four rhombic dodecahedra can be dissected to a tetrahedron and a truncated tetrahedron.
Since 4 rhombic dodecahedra can be dissected to 8 cubes, which form a larger cube, this gives another proof for dissections of tetrahedra to a cube.
The similar results follow from space filling by combination of Archimedean solids with octahedral symmetry.
In [4] the following combinations are examined: truncated octahedron [pg. 167], 2 tetrahedra + octahedron [pg. 172], tetrahedron + truncated tetrahedron [pg. 173],2 tetrahedra, cube, rhombicuboctahedron, [pg. 174],
3 cubes + cuboctahedron + rhombicuboctahedron, 3 cubes + truncated octahedron + truncated cuboctahedron [pg. 176], octahedron + cuboctahedron [pg. 177], octahedron + truncated cube [pg. 178], 2 truncated tetrahedra + truncated cube + truncated cuboctahedron,
2 truncated tetrahedra + truncated octahedron + cuboctahedron [pg. 180], 3 octogonal prisms + truncated cuboctahedron fpg. 181],
3 cubes + 3 octogonal prisms + truncated cube + rhombicuboctahedron [pg. 182].
On the basis of space filling and on some other ideas manny dissection results were obtained [2, pg. 240], [3, pgs. 200-207]:
two truncated octahedra to cube, one truncated octahedron to cube, two tetrahedra + octahedron to cube,
two truncated octahedra to a truncated tetrahedron and a tetrahedron, two truncated octahedra to cuboctahedron and an octahedron,...
The authors of [1, pg. 331] produced the table of the Dehn invariant for the non-snub unit edge Archimedean polyhedra. Let have considered the icosahedral part in [18]. Here we present our investigation into octahedral part.
The table of Dehn invariant for the non-snub unit edge Archimedean polyhedra (octahedral part) [1]:

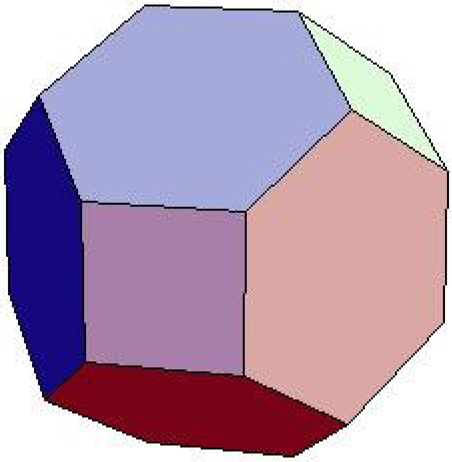
Tetrahedron	-12(3)2
Truncated tetrahedron	12(3)2
Cube	0
Truncated cube	-24(3)2
Octahedron	24(3)2
Truncated octahedron	0
Rhombicuboctahedron	-24(3)2 ???
Cuboctahedron	-24(3)2
Truncated cuboctahedron	0

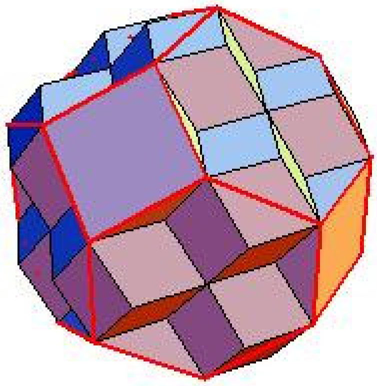
From this table the following combinations with sum 0 exist.

1	Cube
2	Truncated octahedron
3	Truncated cuboctahedron
4	Tetrahedron+truncated tetrahedron
5	Truncated cube+ octahedron
6	Octahedron+ rhombicuboctahedron ???
7	Octahedron+ cuboctahedron
8	2 tetrahedron+octahedron
9	2 truncated tetrahedron+cuboctahedron
10	2 tetrahedron+cube+rhombicuboctahedron
11	3 cube+cuboctahedro+rhombicuboctahedor
12	3 cube+truncated octahedron+truncated cuboctahedron
13	2 truncated tetrahedron+truncated cube+truncated cuboctahedron
14	2 truncated tetrahedron+truncated octahedron+cuboctahedron

The combination 4 was examined in [5,4,2]. Here we shall examine the case of octahedron, cuboctahedron and truncated octahedron. For each of the solids the best "approximate" by rhombic solids is constructed.
In case of truncated octahedron the approximate is complete, that means that the solids are equidecomposable.
In case of the octahedron we have deficit of 8 triangular pyramids (each of which is 1/4 of the tetrahedron);
in case of coboctahedron we have surplus of 8 pyramids. This gives another proof that the cuboctahedron and the octahedron can be dissected to a cube. Since tetrahedron has surplus of 4 pyramids (we count empty set as a rhombic solid), two tetrahedra and the octahedron can be dissected to a cube.
By a rhombic solid we mean a solid composed of rhombic dodecahedra and rhombohedra, where we allow 1/2 of rhombic dodecahedron as well.





References

[1] J. H. Conway, C. Radin, and L. Sadun, On angles whose squared trigonometric functions are rational, Discrete & Computational Geometry, 22 (1999), pages 321-332.
[2] G. N. Frederickson, Dissections: Plane & Fancy, Cambridge U. Press, 1997.
[3] G. N. Frederickson, Hinged Dissections:Swinging & Twisting, Cambridge U. Press, 2002.
[4] R. Williams, The Geometrical Foundation Of Natural Structure, Dover 1972
[5] I. Hafner, T. Zitko, Dissection of rhombic 36-hedron to a tetrahedron and a truncated tetrahedron