Solution of Conway-Radin-Sadun Problem, Summary of Results

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It was proved [1] that it is possible to dissect the icosahedron, dodecahedron, and icosidodecahedron into finitely many pieces that can be reassembled to form a large cube. The authors admit that they have no idea how to perform such dissections. But the solution is quite simple and could be observed on [10]. These solids can be dissected to rhombic triacontahedron and hexecontahedron. The first can be dissected to 10 prolate and 10 oblate rhombohedra and the second to 20 prolate rhombohedra. These prisms can be dissected into pieces and reassembled to a cube according to more or less standard procedure [2].
Te get the hexecontahedron and the triacontahedron on must dissect the icosahedron to 20 pyramids and put them on triangular faces of the icosidodecahedron; excavate 12 pentagonal pyramids from icosidodecahedron and put them on the dodecahedron.

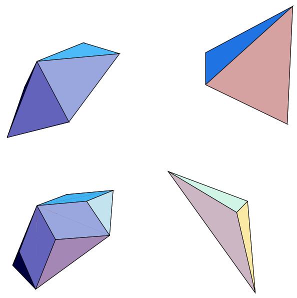
The authors of [1, pg. 331] produced the table of the Dehn invariant for the non-snub unit edge Archimedean polyhedra. Let take only icosahedral part:

Icosahedron	$60(3)_5$
Dodecahedron	$-30(5)_1$
Icosidodecahedron	$-60(3)_5+30(5)_1$
Rhombicosidodecahedron	$60(3)_5-30(5)_1$
Truncated icosahedron	$30(5)_1$
Truncated dodecahedron	$-60(3)_5$
Truncated icosidodecahedron	0

From this table the following combinations with sum 0 exist.

Truncated icosidodecahedron
Icosahedron+ Truncated dodecahedron
Dodecahedron+ Truncated icosahedron
Icosahedron+ Dodecahedron+ Icosidodecahedron
Icosidodecahedron+ Rhombicosidodecahedron
Truncated icosahedron+ Truncated dodecahedron+ Rhombicosidodecahedron

For each this combination we found an equidecomposable combination of rhombic solids. By rhombic solid we mean a polyhedron that consists of prolate and oblate golden rhombohedra. In the solution we use also halves of rhombic dodecahedron of the second kind, which in turn consists of two halves of the rhombohedra. It is obvious, that rhombic solids can be dissected to a cube. The idea is to find the best “approximation” of an Archimedean solid by a rhombic solid. In fact there are basically two solutions depending on the length of edges of the Archimedean solid, which may be shorter or larger diagonal of the rhombus (although the larger edges ara also possible [7]).
Each Archimedean solid has a surplus or a deficit relative its approximation. We found examples where only two types of solids appear as differences:

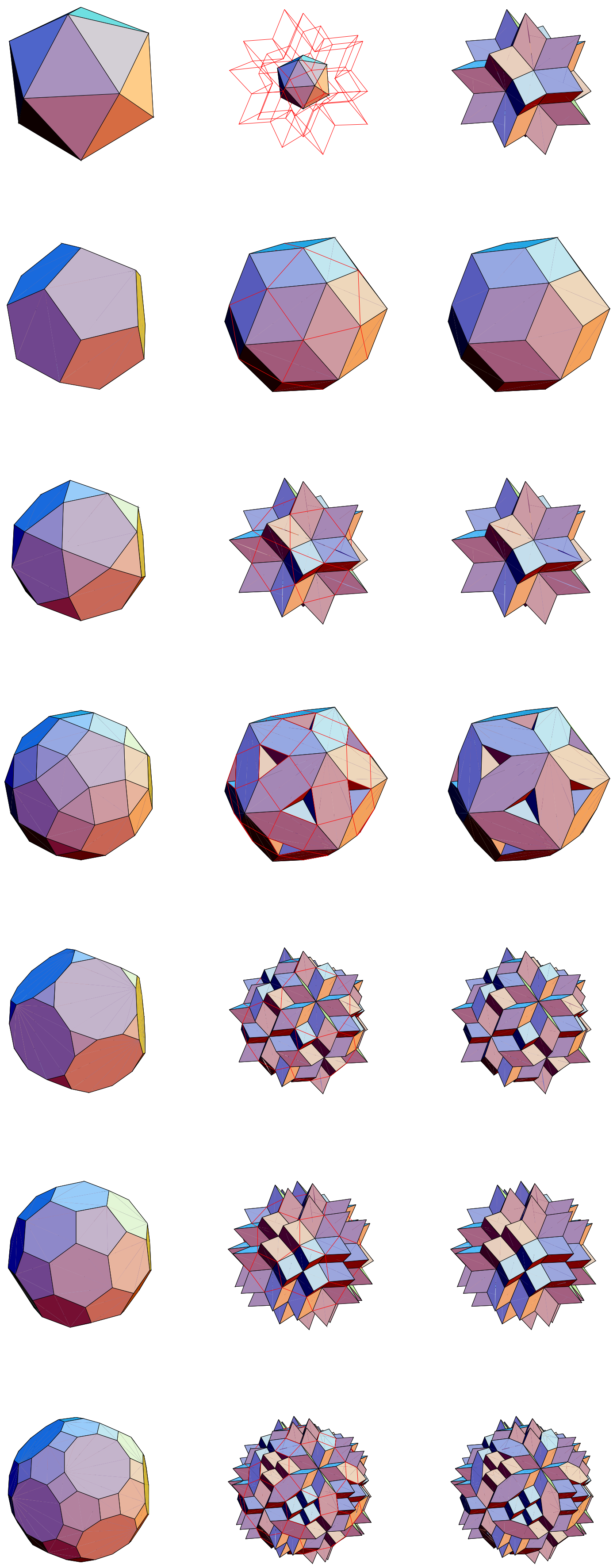


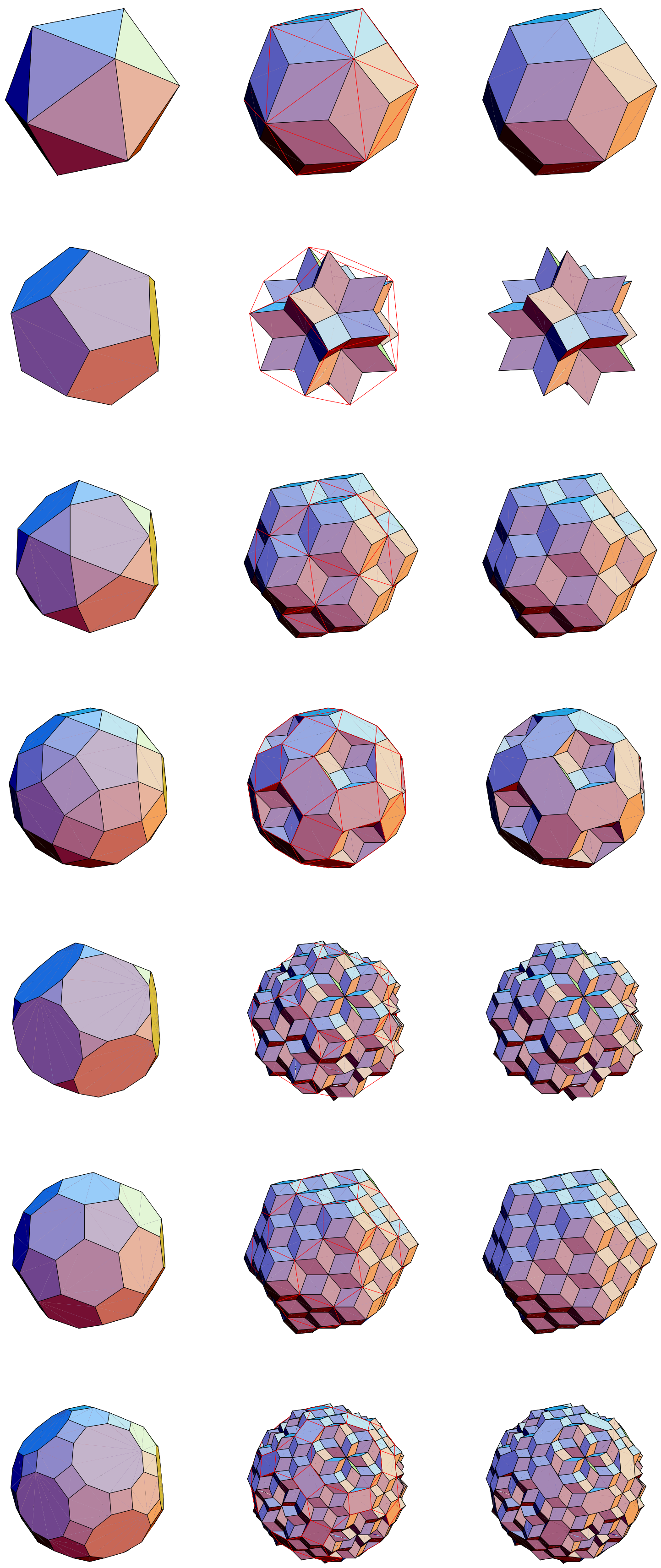
For the first family they are a pentagonal pyramid (P5) that can be cut off triacontahedron and triangular pyramid, that is 1/20 of icosahedron (P3A). For the second family they are pentagonal cup (or cap)(C5) and a triangular pyramid cut off oblate rhombohedron (P3B).
Archimedean solids and their approximations are given by following two tables (we treated icosahedron in the first table in relation to empty set):

Solid	Defficit	Surplus
Icosahedron		20P3A
Dodecahedron	12P5	
Icosidodecahedron	20P3A	12P5
Rhombicosidodecahedron	12P5	20P3A
Truncated icosahedron		12P5
Truncated dodecahedron	20P3A	
Truncated icosidodecahedron		

Solid	Defficit	Surplus
Icosahedron	20P3B	
Dodecahedron		12C5
Icosidodecahedron	12C5	20P3B
Rhombicosidodecahedron	20P3B	12C5
Truncated icosahedron	12C5	
Truncated dodecahedron		20P3B
Truncated icosidodecahedron		

Note certain duality in the tables. For each of 6 combination and 2 sizes we observe equal decomposition.





References

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[3] A dissection of two rhombic dodecahedra of the second kind to a cube
[4] A dissection of quarter of rhombic dodecahedron of the second kind to a cube
[5] Constructions of rhombic hexecontahedra
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[7] Families of golden rhombic polyhedra
[8] From dissection of the cube to space filling with prolate rhombohedra and rhombic dodecahedra of the second kind
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[10] Relations among rhombic, Platonic and Archimedean solids
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[12] Gallery of rhombic polyhedra (LiveGraphics-3D)
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[13]Solution of Conway-Radin-Sadun Problem
[14] Dissection of rhombic 210-hedron to truncated dodecahedron and icosahedron
[15] Dissection of rhombic hexecontahedron and 270-hedron to dodecahedron and truncated icosahedron
[16] Dissection of rhombic solids to icosidodecahedron and rhombicosidodecahedron
[17] Dissection of a solid to truncated icosidodecahedron